

# B $_{\pi}$ -CHARACTERS AND QUOTIENTS

MARK L. LEWIS

ABSTRACT. Let  $\pi$  be a set of primes, and let  $G$  be a finite  $\pi$ -separable group. We consider the Isaacs B $_{\pi}$ -characters. We show that if  $N$  is a normal subgroup of  $G$ , then  $B_{\pi}(G/N) = \text{Irr}(G/N) \cap B_{\pi}(G)$ .

All groups in this paper are finite. Let  $\pi$  be a set of primes and let  $G$  be a  $\pi$ -separable group. In [3], Isaacs defined the subset  $B_{\pi}(G)$  of  $\text{Irr}(G)$ . In this note we are going to prove the following:

**Theorem 1.** *Suppose  $\pi$  is a set of primes and  $G$  is a  $\pi$ -separable group. If  $N$  is a normal subgroup of  $G$ , then  $B_{\pi}(G/N) = \text{Irr}(G/N) \cap B_{\pi}(G)$ .*

Following Gajendragadkar in [1], we say that a character  $\chi \in \text{Irr}(G)$  is  $\pi$ -special if  $\chi(1)$  is a  $\pi$ -number and for every subnormal subgroup  $S$  of  $G$ , the irreducible constituents of  $\chi_S$  have determinantal order that is a  $\pi$ -number. In Proposition 7.1 of [1], Gajendragadkar proved that if  $\alpha, \beta \in \text{Irr}(G)$  are characters so that  $\alpha$  is  $\pi$ -special and  $\beta$  is  $\pi'$ -special, then  $\alpha\beta$  is irreducible, and this factorization is unique. I.e., if  $\alpha\beta = \alpha'\beta'$  where  $\alpha'$  is  $\pi$ -special and  $\beta'$  is  $\pi'$ -special, then  $\alpha = \alpha'$  and  $\beta = \beta'$ .

Using [3], we say that  $\chi \in \text{Irr}(G)$  is  $\pi$ -factored if there exists  $\pi$ -special  $\alpha$  and  $\pi'$ -special  $\beta$  so that  $\chi = \alpha\beta$ . The following lemma regarding the kernels of  $\pi$ -factored characters is key to our argument.

**Lemma 2.** *Suppose  $\pi$  is a set of primes and  $G$  is a  $\pi$ -separable group. If  $\chi \in \text{Irr}(G)$  satisfies  $\chi = \alpha\beta$  where  $\alpha$  is  $\pi$ -special and  $\beta$  is  $\pi'$ -special, then  $\ker(\chi) = \ker(\alpha) \cap \ker(\beta)$ .*

*Proof.* It is obvious that  $\ker(\alpha) \cap \ker(\beta) \leq \ker(\chi) = K$ . We need to show that  $K \leq \ker(\alpha) \cap \ker(\beta)$ . We first claim that  $K \leq Z(\alpha) \cap Z(\beta)$ . Suppose  $g \in K$ . Then  $\alpha(g)\beta(g) = \chi(g) = \chi(1) = \alpha(1)\beta(1)$ . Hence,  $\alpha(1)\beta(1) = |\alpha(g)\beta(g)| = |\alpha(g)||\beta(g)|$ . By Lemma 2.15(c) of [2], we know that  $|\alpha(g)| \leq \alpha(1)$  and  $|\beta(g)| \leq \beta(1)$ . The previous equality implies that these inequalities must be equalities, so  $g \in Z(\alpha)$  and  $g \in Z(\beta)$ . This proves the claim.

By Lemma 2.27 (c) of [2], we see that  $\alpha_K = \alpha(1)\mu$  and  $\beta_K = \beta(1)\nu$  for linear characters  $\mu$  and  $\nu$  in  $\text{Irr}(K)$ . Because  $\alpha$  is  $\pi$ -special,  $\mu$  must have  $\pi$ -order and because  $\beta$  is  $\pi'$ -special,  $\nu$  must have  $\pi'$ -order. For  $g \in K$ , this implies that  $\mu(g)$  is a  $\pi$  root of unity and  $\nu(g)$  is a  $\pi'$ -root of unity. We have  $\alpha(1)\beta(1) = \chi(1) = \chi(g) = \alpha(g)\beta(g) = \alpha(1)\mu(g)\beta(1)\nu(g)$ . This

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implies that  $\nu(g)\mu(g) = 1$ . The only way that the product of a  $\pi$ -root of unity and a  $\pi'$ -root can equal 1 is if they are both 1. I.e., we must have  $\mu(g) = \nu(g) = 1$ . This implies that  $\alpha(1) = \alpha(g)$  and  $\beta(1) = \beta(g)$ . Therefore,  $g \in \ker(\alpha) \cap \ker(\beta)$  as desired.  $\square$

We continue to let  $\pi$  be a set of primes and  $G$  be a  $\pi$ -separable group, and we fix  $\chi \in \text{Irr}(G)$ . We say that  $(S, \sigma)$  is a *subnormal pair* for  $\chi$  if  $S$  is a subnormal subgroup of  $G$ ,  $\sigma$  is an irreducible constituent of  $\chi_S$ . In addition, we say that  $(S, \sigma)$  is  $\pi$ -factored if  $\sigma$  is  $\pi$ -factored. We can define a partial ordering on the subnormal pairs for  $\chi$  by  $(S, \sigma) \leq (T, \tau)$  if  $S \leq T$  and  $\sigma$  is a constituent of  $\tau_S$ .

Notice that  $(1, 1_1)$  is a  $\pi$ -factored subnormal pair for  $\chi$ , so there exists a maximal  $\pi$ -factored subnormal pair for  $\chi$  with respect to the partial ordering. It is shown in Theorem 3.2 of [3] that the set of maximal  $\pi$ -factored subnormal pairs for  $\chi$  are conjugate in  $G$ . Let  $(S, \sigma)$  be a maximal  $\pi$ -factored subnormal pair for  $\chi$ , and let  $T$  be the stabilizer of  $(S, \sigma)$  in  $G$ . It is shown in Theorem 4.4 of [3] that there is a unique  $\tau \in \text{Irr}(T \mid \sigma)$  so that  $\tau^G = \chi$ .

We can now define the  $\pi$ -nucleus for  $\chi$ . If  $\chi$  is  $\pi$ -factored, then  $(G, \chi)$  is the nucleus for  $\chi$ . If  $\chi$  is not  $\pi$ -factored, then let  $(S, \sigma)$  be a maximal  $\pi$ -factored subnormal pair for  $\chi$ . Let  $T$  be the stabilizer of  $(S, \sigma)$  in  $G$ , and let  $\tau \in \text{Irr}(T \mid \sigma)$  so that  $\tau^G = \chi$ . By Lemma 4.5 of [3], we know that  $T < G$ , so we can inductively define the  $\pi$ -nucleus of  $\chi$  to be the  $\pi$ -nucleus of  $\tau$ . Because the maximal  $\pi$ -factored subnormal pairs are all conjugate, it follows that the  $\pi$ -nucleus for  $\chi$  is well-defined up to conjugacy. (See the argument on page 108 of [3].)

If  $(X, \eta)$  is a  $\pi$ -nucleus for  $\chi$ , then it is not difficult to see that  $\eta$  must be  $\pi$ -factored. As defined in Definition 5.1 of [3], we say that  $\chi \in B_\pi(G)$  if and only if  $\eta$  is  $\pi$ -special where  $(X, \eta)$  is a  $\pi$ -nucleus for  $\chi$ .

**Lemma 3.** *Let  $\pi$  be a set of primes and let  $G$  be a  $\pi$ -separable group. Suppose that  $N$  is a normal subgroup of  $G$ . If  $\chi \in \text{Irr}(G/N)$  has  $\pi$ -nucleus  $(X, \eta)$ , then  $(X/N, \eta)$  is a  $\pi$ -nucleus for  $\chi$  viewed as character in  $\text{Irr}(G/N)$ .*

*Proof.* If  $(X, \eta) = (G, \chi)$ , then this is obvious. Thus, we may assume that  $X < G$ . Let  $(S, \sigma)$  be a maximal  $\pi$ -factored subnormal pair for  $\chi$  with stabilizer  $T$  and character  $\tau \in \text{Irr}(T \mid \sigma)$  so that  $\tau^G = \chi$  and  $(X, \eta)$  is a  $\pi$ -nucleus for  $\tau$ . Notice that  $(N, 1_N)$  is a  $\pi$ -factored subnormal pair for  $\chi$ , so it is contained in a maximal such pair. Since  $N$  is normal, this implies that  $N \leq S$ . Because  $\sigma$  is a constituent of  $\chi_S$ , we see that  $N \leq \ker(\sigma)$ . By Lemma 2, we see that  $\sigma$  is  $\pi$ -factored as a character in  $\text{Irr}(S/N)$ . Notice that  $(S/N, \sigma) \leq (S^*/N, \sigma^*)$  if and only if  $(S, \sigma) \leq (S^*, \sigma^*)$ , and by Lemma 2,  $\sigma^*$  is  $\pi$ -factored in  $\text{Irr}(S^*/N)$  if and only if it is  $\pi$ -factored in  $\text{Irr}(S^*)$ . Therefore,  $(S/N, \sigma)$  must be a maximal  $\pi$ -factored subnormal pair for  $\chi$  viewed as a character in  $\text{Irr}(G/N)$ . It is immediate that  $T/N$  will be the stabilizer for  $(S/N, \sigma)$  in  $G/N$  and that  $\tau$  is the unique character in  $\text{Irr}(T/N \mid \sigma)$  that induces  $\chi$ . By induction,  $(X/N, \eta)$  will be the  $\pi$ -nucleus for  $\tau$  viewed as a

character of  $\text{Irr}(T/N)$ , and thus,  $(X/N, \eta)$  will be the  $\pi$ -nucleus for  $\chi$  viewed as a character of  $\text{Irr}(G/N)$ .  $\square$

We are now ready to prove Theorem 1.

*Proof of Theorem 1.* Note that  $B_\pi(G/N) \subseteq \text{Irr}(G/N)$ . Hence, it suffices to show for  $\chi \in \text{Irr}(G/N)$  that  $\chi \in B_\pi(G)$  if and only if  $\chi \in B_\pi(G/N)$ . Suppose  $\chi \in \text{Irr}(G/N)$ . Let  $(X, \eta)$  be a  $\pi$ -nucleus for  $\chi$ . By Lemma 3,  $(X/N, \eta)$  is a nucleus for  $\chi$  viewed as a character of  $\text{Irr}(G/N)$ . Note that  $\eta$  is  $\pi$ -special as character in  $\text{Irr}(X)$  if and only if it is  $\pi$ -special viewed as a character of  $\text{Irr}(X/N)$ . We know that  $\chi \in B_\pi(G)$  if and only if  $\eta$  is  $\pi$ -special character of  $\text{Irr}(X)$  and  $\chi \in B_\pi(G/N)$  if and only if  $\eta$  is  $\pi$ -special as a character of  $\text{Irr}(X/N)$ . Since we saw that these are equivalent, this proves the theorem.  $\square$

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DEPARTMENT OF MATHEMATICAL SCIENCES, KENT STATE UNIVERSITY, KENT, OH 44242

*E-mail address:* lewis@math.kent.edu